

Bifurcation curves of a diffusive logistic equation with harvesting orthogonal to the first eigenfunction*

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Abstract

We study the global bifurcation curves of a diffusive logistic equation, when harvesting is orthogonal to the first eigenfunction of the Laplacian, for values of the linear growth up to $\lambda_2 + \delta$, examining in detail their behavior as the linear growth rate crosses the first two eigenvalues. We observe some new behavior with regard to earlier works concerning this equation. Namely, the bifurcation curves suffer a transformation at λ_1 , they are compact above λ_1 , there are precisely two families of degenerate solutions with Morse index equal to zero, and the whole set of solutions below λ_2 is not a two dimensional manifold.

1 Introduction

This paper concerns the study of logistic equations of the form

$$-\Delta u = au - f(u) - ch, \quad (1)$$

in a smooth bounded domain $\Omega \in \mathbb{R}^N$, with $N \geq 1$. We are interested in weak solutions belonging to the space

$$\mathcal{H} = \{u \in W^{2,p}(\Omega) : u = 0 \text{ on } \partial\Omega\},$$

for some fixed $p > N$. Let λ_1 and λ_2 be the first and second eigenvalues of the Dirichlet Laplacian on Ω , respectively. We denote by ϕ the first eigenfunction satisfying $\max_{\Omega} \phi = 1$. We assume that λ_2 is simple, with eigenspace spanned by ψ , and we also normalize the second eigenfunction so $\max_{\Omega} \psi = 1$.

The competition term f is assumed to satisfy the following hypotheses:

- (i) $f \in C^2(\mathbb{R})$.
- (ii) $f(u) = 0$ for $u \leq M$, and $f(u) > 0$ for $u > M$; throughout $M \geq 0$ is fixed.

*2010 Mathematics Subject Classification: 35B32, 35J66, 37B30, 92D25.

Keywords: Bifurcation theory, Morse indices, logistic equation, degenerate solutions.

[†]Email: pgirao@math.ist.utl.pt. Partially supported by the Fundação para a Ciência e a Tecnologia (Portugal) and by project UTAustin/MAT/0035/2008.

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(iii) $f''(u) \geq 0$.

(iv) $\lim_{u \rightarrow +\infty} \frac{f(u)}{u} = +\infty$.

In [13], the authors obtained global bifurcation curves, of positive solutions to (1), for values of the parameter a in a right neighborhood of λ_1 , when $f(u) = u^2$ and h is a positive function.

In [10], the first author generalized the results of [13] to competition terms satisfying (i)-(iv), and studied the bifurcation curves, of sign changing solutions, for a up to $\lambda_2 + \delta$, for some $\delta > 0$. This was also done under the assumption that h was positive a.e. in Ω , a hypothesis which was used in the proof, although, as noted in [10], in a right neighborhood of λ_1 , one may relax the requirement on h to $\int_{\Omega} h \phi dx \neq 0$.

In this paper, we analyze the situation when the harvesting function h , which in our case might be more appropriately called harvesting and plantation function, is orthogonal to the first eigenfunction of the Laplacian. The biological interpretation gives our context but should be taken with care, as it breaks down in several circumstances, for instance, if the solutions become negative. Our motivation is mathematical, we are forced to provide new arguments, and we suspected the geometry of the problem would be different from the one in [10]. Indeed, it turns out that the bifurcation curves suffer a complete transformation when the parameter a crosses the first eigenvalue. We examine in detail the way in which this change occurs. When seen in the (a, u, c) space, the set of solutions of (1) between λ_1 and λ_2 has the shape of a piece of a paraboloid, with a flat bottom at $a = \lambda_1$. A 2-dimensional space of solutions is attached to this bottom at $a = \lambda_1$, along a segment, and lies in the region $a \leq \lambda_1$. The whole set of solutions below λ_2 is not a two-dimensional manifold. Therefore we find a richer behavior regarding this equation than in the earlier works. Also, in contrast to the bifurcation curves obtained in the previous papers, our curves turn out to be compact above λ_1 , and, instead of one, we get two families of degenerate solutions with Morse index equal to zero above λ_1 .

Specifically, we assume:

(a) $h \in L^{\infty}(\Omega)$.

(b) $\int_{\Omega} h \phi dx = 0$.

(c) $\int_{\Omega} h \psi dx \neq 0$.

Hypothesis (c) also appears in [10]. Our main results are Theorems 3.2, 4.8, 4.11, 4.12, 5.1 and 5.2. The proofs involve bifurcation methods ([7, 8]), a blow up argument, the Morse indices, and a careful choice of coordinates at each step. In particular, around λ_1 we decompose the space \mathcal{H} as in [3]. In the end, we obtain a complete picture of the set of solutions for the parameter a up to $\lambda_2 + \delta$.

For other works related to logistic equations with harvesting we refer the reader to [5, 11, 14].

This paper is organized as follows: We treat successively the cases where the linear growth parameter a is equal to λ_1 (Section 2), below λ_1 (Section 3), between λ_1 and λ_2 (Section 4), and greater than or equal to λ_2 (Section 5).

Acknowledgments. This research was conducted during a stay of the second author at IST. She is grateful for the pleasant atmosphere.

2 Linear growth a equal to λ_1

Assume $(\lambda_1, u, c) \in \mathbb{R} \times \mathcal{H} \times \mathbb{R}$ is a solution of (1) with $a = \lambda_1$. Multiplying both sides of (1) by ϕ and integrating, taking into account **(b)** and $-\Delta\phi = \lambda_1\phi$, we deduce that $\int f(u)\phi dx = 0$. When the region of integration is omitted it is understood to be Ω . Because ϕ is positive in Ω and $f(u)$ is continuous in Ω , we get that $f(u) \equiv 0$. This means $u \leq M$ by **(ii)**. Therefore, for $a = \lambda_1$ the solutions of (1) are those of the linear problem

$$-\Delta u = \lambda_1 u - ch,$$

i.e. are of the form (λ_1, u, c) , where $u = t\phi + c(\Delta + \lambda_1)^{-1}h$, with

$$(t, c) \in \Lambda := \{(t, c) \in \mathbb{R}^2 : t\phi + c(\Delta + \lambda_1)^{-1}h \leq M\}. \quad (2)$$

Thus, there is a bijection between the set of solutions of (1) for $a = \lambda_1$ and Λ , given by $(\lambda_1, t\phi + c(\Delta + \lambda_1)^{-1}h, c) \leftrightarrow (t, c)$. The set Λ is closed and convex.

Let

$$T := \sup\{t : \text{there exists } c \text{ such that } (t, c) \in \Lambda\}. \quad (3)$$

Taking $c = 0$ and using the normalization $\max_{\Omega} \phi = 1$, $(M, 0) \in \Lambda$ and so $T \geq M$. The value of T is finite. Indeed, h and $(\Delta + \lambda_1)^{-1}h$ are orthogonal to ϕ and so $(\Delta + \lambda_1)^{-1}h$ changes sign. Let Ω_+ be the set where $(\Delta + \lambda_1)^{-1}h$ is positive and let Ω_- be the set where $(\Delta + \lambda_1)^{-1}h$ is negative. Suppose that $c \geq 0$; then $M - c(\Delta + \lambda_1)^{-1}h \leq M$ on Ω_+ , and so if $(t, c) \in \Lambda$, then $t\phi \leq M$ on Ω_+ . Suppose $c < 0$; then $M - c(\Delta + \lambda_1)^{-1}h \leq M$ on Ω_- , and so if $(t, c) \in \Lambda$, then $t\phi \leq M$ on Ω_- . In any case, $c \geq 0$ or $c < 0$, $t\phi \leq M$ either on Ω_+ or on Ω_- . We conclude that $T < +\infty$ as asserted. The value T is a maximum.

To characterize parts of the boundary of Λ , we define two functions, $c_{\lambda_1}^-$ and $c_{\lambda_1}^+$, in the interval $] -\infty, T]$, by

$$c_{\lambda_1}^-(t) := \min_{(t,c) \in \Lambda} c \quad \text{and} \quad c_{\lambda_1}^+(t) := \max_{(t,c) \in \Lambda} c. \quad (4)$$

Clearly, $c_{\lambda_1}^-(T) \leq c_{\lambda_1}^+(T)$. Notice that

$$\lim_{t \rightarrow -\infty} c_{\lambda_1}^-(t) = -\infty \quad \text{and} \quad \lim_{t \rightarrow -\infty} c_{\lambda_1}^+(t) = +\infty, \quad (5)$$

since, when $t \rightarrow -\infty$, denoting by ν the unit outward normal to Ω , using Hopf's Lemma, we have $\frac{\partial}{\partial \nu}(M - t\phi) = -t \frac{\partial \phi}{\partial \nu} \geq -t \max_{\partial \Omega} \frac{\partial \phi}{\partial \nu} \rightarrow -\infty$, and $M - t\phi \rightarrow +\infty$ uniformly in each compact subset of Ω . Therefore, because $(\Delta + \lambda_1)^{-1}h$ belongs to $C^1(\overline{\Omega})$, as t goes to $-\infty$, it is possible to guarantee that $c(\Delta + \lambda_1)^{-1}h \leq M - t\phi$ for larger and larger values of $|c|$. This establishes (5). From (5) and the fact that Λ is convex, it follows that $c_{\lambda_1}^-$

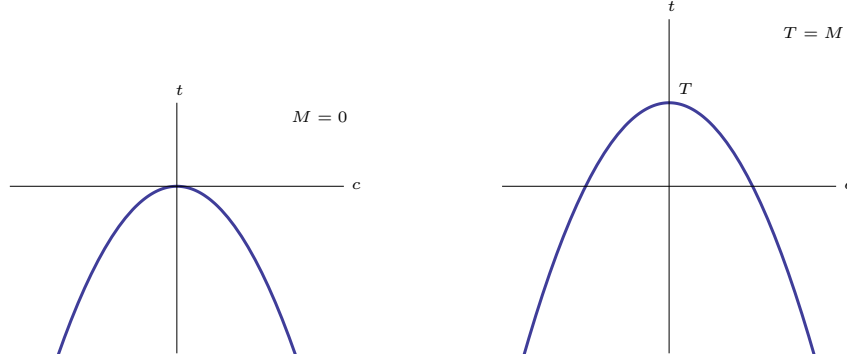


Figure 1: The boundary of the set Λ when $M = 0$ and when $T = M$.

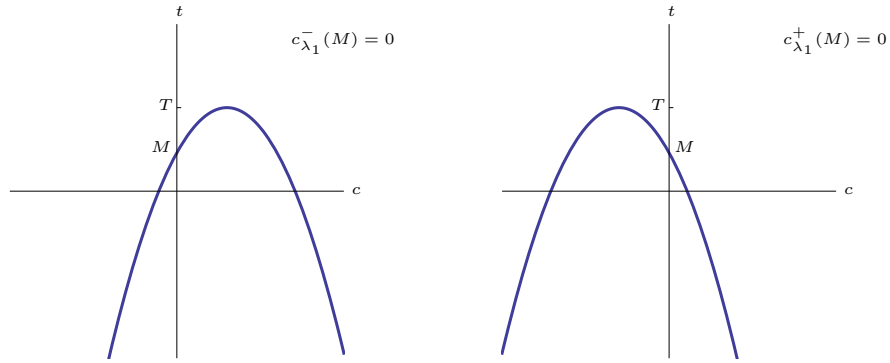


Figure 2: The boundary of the set Λ when $T > M$.

is convex, continuous and strictly increasing. Similarly, $c_{\lambda_1}^+$ is concave, continuous and strictly decreasing.

If $M = 0$, then $T = 0$ and $c_{\lambda_1}^-(0) = c_{\lambda_1}^+(0) = 0$, because the function $(\Delta + \lambda_1)^{-1}h$ changes sign. One can check with specific examples, that it might happen that $T = M$. In such a situation $c_{\lambda_1}^-(0) \leq 0 \leq c_{\lambda_1}^+(0)$ because $(M, 0) \in \Lambda$ and so $0 \geq c_{\lambda_1}^-(M) \geq c_{\lambda_1}^-(0)$, $0 \leq c_{\lambda_1}^+(M) \leq c_{\lambda_1}^+(0)$ (see Figure 1). On the other hand, if $T > M$, then either $c_{\lambda_1}^-(M) = 0$ or $c_{\lambda_1}^+(M) = 0$ (see Figure 2). Indeed, take $M < t < T$ and $(t, c) \in \Lambda$. We have that $c(\Delta + \lambda_1)^{-1}h \leq M - t\phi$. The function $M - t\phi$ is negative in an open subset of Ω . Therefore, $\{c : (t, c) \in \Lambda\} \subset \mathbb{R}^-$ or $\{c : (t, c) \in \Lambda\} \subset \mathbb{R}^+$, as the function $c(\Delta + \lambda_1)^{-1}h$ cannot vanish at any point of that open subset of Ω . This shows that $c_{\lambda_1}^-(t) > 0$ or $c_{\lambda_1}^+(t) < 0$, for $M < t < T$. Passing to the limit as $t \searrow M$, $c_{\lambda_1}^-(M) \geq 0$ or $c_{\lambda_1}^+(M) \leq 0$. On the other hand, since $(M, 0) \in \Lambda$, it holds that $c_{\lambda_1}^-(M) \leq 0 \leq c_{\lambda_1}^+(M)$. Therefore, $c_{\lambda_1}^-(M) = 0$ or $c_{\lambda_1}^+(M) = 0$, as claimed. In any of the three possible cases, $M = T = 0$, $0 < M = T$ and $0 < M < T$, we have that

$$c_{*,\lambda_1}^- := c_{\lambda_1}^-(0) \leq 0 \quad \text{and} \quad c_{*,\lambda_1}^+ := c_{\lambda_1}^+(0) \geq 0. \quad (6)$$

We can describe the solutions (a, u, c) of (1) in a neighborhood of $(\lambda_1, t_0\phi + c_0(\Delta +$

$\lambda_1)^{-1}h, c_0)$ where, of course, $(t_0, c_0) \in \Lambda$. We define

$$\mathcal{R} := \{y \in \mathcal{H} : \int y \phi dx = 0\}.$$

Lemma 2.1. *Let $(t_0, c_0) \in \Lambda$ with $t_0 \neq 0$. There exists a neighborhood $U \subset \mathbb{R} \times \mathcal{H} \times \mathbb{R}$ of $(\lambda_1, t_0\phi + c_0(\Delta + \lambda_1)^{-1}h, c_0)$ such that the solutions of (1) in U are a C^1 manifold which can be parametrized, in a neighborhood V of (t_0, c_0) , by $(t, c) \mapsto (a(t, c), t\phi + y(t, c), c)$. Here $y \in \mathcal{R}$. For $t > 0$ we have that $a(t, c) \geq \lambda_1$, whereas for $t < 0$ it holds that $a(t, c) \leq \lambda_1$.*

Proof. This lemma is a direct consequence of the Implicit Function Theorem applied to the function $g : \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times \mathbb{R} \rightarrow L^p(\Omega)$, defined by $g(a, t, y, c) = \Delta(t\phi + y) + a(t\phi + y) - f(t\phi + y) - ch$, at the point $(\lambda_1, t_0, c_0(\Delta + \lambda_1)^{-1}h, c_0) \leftrightarrow (t_0, c_0)$ with $t_0 \neq 0$. Let $(\alpha, z) \in \mathbb{R} \times \mathcal{R}$. The derivative of g with respect to (a, y) in the direction of (α, z) , $Dg_{(a,y)}(\alpha, z)$, computed at the point $(\lambda_1, t_0, c_0(\Delta + \lambda_1)^{-1}h, c_0)$, is

$$\begin{aligned} g_a \alpha + g_y z &= \alpha(t\phi + y) + \Delta z + az - f'(t\phi + y)z \\ &= \alpha[t_0\phi + c_0(\Delta + \lambda_1)^{-1}h] + \Delta z + \lambda_1 z. \end{aligned}$$

We used the fact that f' vanishes below M . We check that this derivative is injective. Setting the previous derivative equal to zero, and looking at the ϕ component of the right hand side, we first get that $\alpha = 0$, because $(\Delta + \lambda_1)^{-1}h$ has no ϕ component and $t_0 \neq 0$. Thus $\Delta z + \lambda_1 z = 0$. Since $z \in \mathcal{R}$, it follows that $z = 0$. This proves injectivity. It is easy to check that the derivative is also surjective. So the derivative, computed at the point $(\lambda_1, t_0, c_0(\Delta + \lambda_1)^{-1}h, c_0)$ with $t_0 \neq 0$, is a homeomorphism from $\mathbb{R} \times \mathcal{R}$ to $L^p(\Omega)$. The statement about the sign of a follows from the equality

$$(a - \lambda_1)t \int \phi^2 dx = \int f(u)\phi dx, \quad (7)$$

where $u = t\phi + y$. □

For use in the next sections, where we consider values of a different from λ_1 , we make the following

Remark 2.2. *Let (a_n, u_n, c_n) be a sequence of solutions of (1) with (a_n) and (c_n) bounded. Then (u_n) is uniformly bounded above. The same conclusion follows if, instead of assuming (c_n) bounded, we suppose that $(\frac{c_n}{\max u_n})$ is bounded.*

Indeed, denoting by x_n a point of maximum for u_n , clearly

$$a_n u_n(x_n) - f(u_n(x_n)) - c_n h(x_n) \geq 0. \quad (8)$$

Admitting that $u_n(x_n) \rightarrow +\infty$, from

$$\frac{f(u_n(x_n))}{u_n(x_n)} \leq a_n - \frac{c_n}{u_n(x_n)} h(x_n),$$

whose left hand side is bounded, we contradict hypothesis **(iv)**.

3 Linear growth a below λ_1

In this section we are going to analyze the case $a < \lambda_1$. We observe that, for c fixed, there exists a unique solution of (1). Indeed, to find the solution one just has to minimize the coercive (since $a < \lambda_1$ and $F(u) := \int_0^u f(s) ds$ is positive) functional

$$\int \left[\frac{1}{2}(|\nabla u|^2 - au^2) + F(u) + chu \right] dx$$

on $H_0^1(\Omega)$. By Remark 2.2, if (a, u, c) is a solution, $\text{ess sup } u$ is finite. As $f(u)$ only depends on the positive part of u , $f(u)$ belongs to $L^\infty(\Omega)$. By elliptic regularity theory (see [9]), u belongs to \mathcal{H} . The solution is nondegenerate. In fact, suppose that $v \in \mathcal{H}$ is such that

$$-\Delta v - av + f'(u)v = 0.$$

Multiplying both sides of this equation by v and integrating over Ω , we get that

$$\begin{aligned} 0 &= \int [|\nabla v|^2 - av^2 + f'(u)v^2] dx \\ &\geq \int [|\nabla v|^2 - av^2] dx \\ &\geq (\lambda_1 - a) \int v^2 dx \end{aligned}$$

Thus $v = 0$ and the solution is nondegenerate. Thus, for a fixed a the set of solutions of (1) is a one dimensional C^1 manifold in $\{a\} \times \mathcal{H} \times \mathbb{R}$, that can be parametrized by $c \mapsto (a, u_a(c), c)$. This follows from the Implicit Function Theorem applied to the function $G : \mathbb{R} \times \mathcal{H} \times \mathbb{R} \rightarrow L^p(\Omega)$, defined by

$$G(a, u, c) = \Delta u + au - f(u) - ch.$$

Observe that, for $v \in \mathcal{H}$, the derivative $DG_{(u)}(v)$ of G with respect to u in the direction of v , computed at a solution (a, u, c) , is $DG_{(u)}(v) = \Delta v + av - f'(u)v$.

For $a < \lambda_1$, the component in ϕ of a solution (a, u, c) , $\frac{\int u \phi dx}{\int \phi^2 dx} =: t$, is nonpositive due to (7). We wish to examine the behavior of the solutions as a increases to λ_1 .

Lemma 3.1. *Let c^- , c^+ be some constants such that $c^- < c_{*,\lambda_1}^-$ and $c^+ > c_{*,\lambda_1}^+$, with c_{*,λ_1}^- , c_{*,λ_1}^+ given in (6), and let $\hat{t} < 0$. There exists $\delta > 0$ such that for all $\lambda_1 - \delta < a < \lambda_1$ and (a, u, c) solution of (1), with $c^- \leq c \leq c^+$, we have that $t > \tau_{\hat{t}}(c)$ with*

$$\tau_{\hat{t}}(c) := \begin{cases} \min\{(c_{\lambda_1}^-)^{-1}(c), \hat{t}\} & \text{if } c^- \leq c \leq c_{*,\lambda_1}^-, \\ \hat{t} & \text{if } c_{*,\lambda_1}^- \leq c \leq c_{*,\lambda_1}^+, \\ \min\{\hat{t}, (c_{\lambda_1}^+)^{-1}(c)\} & \text{if } c_{*,\lambda_1}^+ \leq c \leq c^+ \end{cases} \quad (9)$$

(see Figure 3).

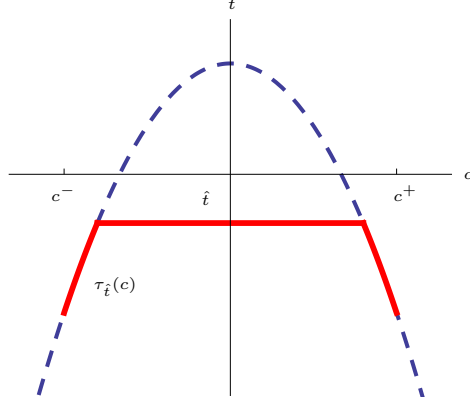


Figure 3: The graph of $\tau_{\hat{t}}$.

Proof. We argue by contradiction. Let $a_n \nearrow \lambda_1$ be such that (a_n, u_n, c_n) is a solution of (1), with $c^- \leq c_n \leq c^+$ and $t_n := \frac{\int u_n \phi dx}{\int \phi^2 dx} \leq \tau_{\hat{t}}(c_n)$. As remarked, for each fixed $a < \lambda_1$, the map $c \mapsto u_a(c)$ is, in particular, continuous, and $u_a(0) = 0$. Note that t is a continuous function of u , which itself is a continuous function of c . Thus, taking into account $t(c_n) = t_n \leq \tau_{\hat{t}}(c_n) < t(0) = 0$ and using the Intermediate Value Theorem for each fixed a , without loss of generality, by changing c_n (and thus u_n), we may assume that

$$t_n = \tau_{\hat{t}}(c_n). \quad (10)$$

In addition, we may suppose that $c_n \rightarrow c_0$. This implies that $t_n \rightarrow t_0$.

Let $y_n = u_n - t_n \phi$. Multiplying (1) by y_n and integrating over Ω ,

$$\int |\nabla y_n|^2 dx = a_n \int y_n^2 dx - \int f(t_n \phi + y_n) y_n dx - c_n \int h y_n dx.$$

We observe that the second term in the right hand side is nonpositive since

$$- \int f(t_n \phi + y_n) y_n dx = - \int f(t_n \phi + y_n) (t_n \phi + y_n) dx + t_n \int f(t_n \phi + y_n) \phi dx. \quad (11)$$

Thus, we have that

$$\int |\nabla y_n|^2 dx \leq a_n \int y_n^2 dx - c_n \int h y_n dx,$$

which, together with

$$\int |\nabla y_n|^2 dx \geq \lambda_2 \int y_n^2 dx,$$

implies first that (y_n) is bounded in $L^2(\Omega)$ and then that (y_n) is bounded in $H_0^1(\Omega)$. We may assume that $y_n \rightarrow y_0$ in $L^2(\Omega)$ and a.e. in Ω . By Remark 2.2, $(\text{ess sup } u_n)$ is uniformly bounded. Letting $u_0 = t_0 \phi + y_0$, from the Dominated Convergence Theorem it follows that $f(u_n) \rightarrow f(u_0)$ in $L^p(\Omega)$. Using equation (1) and elliptic regularity theory (see [9]), $y_n \rightarrow y_0$ in \mathcal{H} .

From the previous paragraph, the limit (λ_1, u_0, c_0) satisfies equation (1) and, from (10), $t_0 = \tau_{\hat{t}}(c_0)$. We use Lemma 2.1 at the point (t_0, c_0) . The solutions of (1) in a neighborhood of the image of (t_0, c_0) can be parametrized by $(t, c) \mapsto (a(t, c), t\phi + y(t, c), c)$. From the choice of (a_n, u_n, c_n) , we have that $a(\tau_{\hat{t}}(c_n), c_n) = a(t_n, c_n) = a_n < \lambda_1$.

To obtain the desired contradiction, we show next that $a(\tau_{\hat{t}}(c), c) = \lambda_1$ for any $c \in [c^-, c^+]$. There are three possible cases: (i) $c = c_{\lambda_1}^-(t)$ for some $t < \hat{t}$, (ii) $c = c_{\lambda_1}^+(t)$ for some $t < \hat{t}$ or (iii) $c_{\lambda_1}^-(\hat{t}) \leq c \leq c_{\lambda_1}^+(\hat{t})$. In case (i) $\tau_{\hat{t}}(c) = t$. So the solution $(\lambda_1, t\phi + c_{\lambda_1}^-(t)(\Delta + \lambda_1)^{-1}h, c_{\lambda_1}^-(t))$ can be written as $(\lambda_1, \tau_{\hat{t}}(c)\phi + c(\Delta + \lambda_1)^{-1}h, c)$, which means that $a(\tau_{\hat{t}}(c), c) = \lambda_1$. Similarly in case (ii). In case (iii) $\tau_{\hat{t}}(c) = \hat{t}$. Since $c_{\lambda_1}^-(\hat{t}) \leq c \leq c_{\lambda_1}^+(\hat{t})$, $(\lambda_1, \hat{t}\phi + c(\Delta + \lambda_1)^{-1}h, c)$ is a solution of (1). Therefore $\lambda_1 = a(\hat{t}, c) = a(\tau_{\hat{t}}(c), c)$.

In conclusion, on the one hand $a(\tau_{\hat{t}}(c_n), c_n) < \lambda_1$ and on the other hand $a(\tau_{\hat{t}}(c), c) = \lambda_1$ for any $c \in [c^-, c^+]$. We reached a contradiction. The lemma is proved. \square

We extend the definition of τ in (9) to zero,

$$\tau_0(c) := \begin{cases} (c_{\lambda_1}^-)^{-1}(c) & \text{if } c^- \leq c \leq c_{*,\lambda_1}^-, \\ 0 & \text{if } c_{*,\lambda_1}^- \leq c \leq c_{*,\lambda_1}^+, \\ (c_{\lambda_1}^+)^{-1}(c) & \text{if } c_{*,\lambda_1}^+ \leq c \leq c^+. \end{cases}$$

In the next theorem, we prove that, as $a \nearrow \lambda_1$, the solutions $(a, u_a(c), c)$ converge to

$$(\lambda_1, u_{\lambda_1}(c), c) := (\lambda_1, \tau_0(c)\phi + c(\Delta + \lambda_1)^{-1}h, c). \quad (12)$$

Theorem 3.2. *Let c^-, c^+ be some constants such that $-\infty < c^- < c_{*,\lambda_1}^- \leq c_{*,\lambda_1}^+ < c^+ < \infty$. The solutions $(a, u_a(c), c)$ are such that the functions $c \mapsto u_a(c)$, for $c \in [c^-, c^+]$, converge uniformly in \mathcal{H} to $u_{\lambda_1}(c)$ as $a \nearrow \lambda_1$.*

Proof. Again the proof is by contradiction. We admit that there exists $\delta > 0$, $a_n \nearrow \lambda_1$ and $c_n \in [c^-, c^+]$ such that the corresponding solutions $(a_n, u_{a_n}(c_n), c_n)$ satisfy

$$\|u_{a_n}(c_n) - u_{\lambda_1}(c_n)\|_{\mathcal{H}} \geq \delta. \quad (13)$$

We may assume that $c_n \rightarrow c_0$. Let $\hat{t} < 0$. By Lemma 3.1, for large n , $\tau_{\hat{t}}(c_n) \leq t_n \leq 0$. Modulo a subsequence, $t_n \rightarrow t_0$, where $\tau_{\hat{t}}(c_0) \leq t_0$. Since \hat{t} is arbitrary, $\tau_0(c_0) \leq t_0$. Again arguing as in the proof of Lemma 3.1, $y_n = u_{a_n}(c_n) - t_n\phi \rightarrow y_0$ in \mathcal{H} and $(a_n, u_{a_n}(c_n), c_n)$ converge to $(\lambda_1, u_0, c_0) := (\lambda_1, t_0\phi + y_0, c_0)$ which must be a solution of (1). Using that $\tau_0(c_0) \leq t_0$ and $t_0 \leq 0$, we conclude $t_0 = \tau_0(c_0)$, because there are no solutions of (1) corresponding to a nonnegative t_0 satisfying $\tau_0(c_0) < t_0$. Therefore $(\lambda_1, u_0, c_0) = (\lambda_1, \tau_0(c_0)\phi + y_0, c_0) = (\lambda_1, u_{\lambda_1}(c_0), c_0)$, according to (12) since necessarily $y_0 = c_0(\Delta + \lambda_1)^{-1}h$. This shows that $u_{a_n}(c_n)$ converges to $u_{\lambda_1}(c_0)$ in \mathcal{H} . Examining (12), $c \mapsto u_{\lambda_1}(c)$ is continuous in \mathcal{H} . Passing to the limit in both sides of (13), we get that

$$0 = \|u_{\lambda_1}(c_0) - u_{\lambda_1}(c_0)\|_{\mathcal{H}} \geq \delta.$$

This contradiction proves the theorem. \square

In Figure 4, we plot the curves $c \mapsto (a, t(c), c)$ for a in an interval $]\lambda_1 - \delta, \lambda_1[$, for some small $\delta > 0$. The value of a increases in the vertical direction.

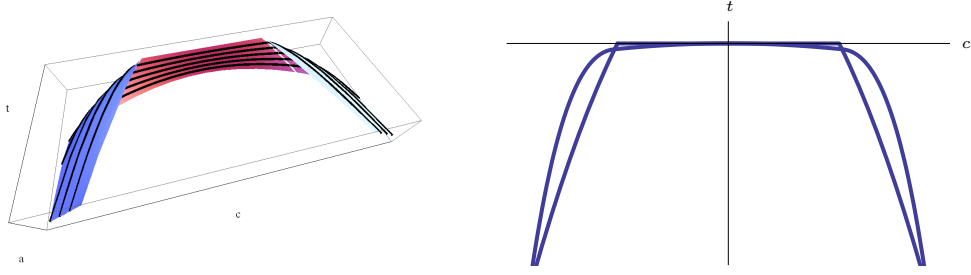


Figure 4: In the left image one can see the surface formed by the curves $c \mapsto (a, t(c), c)$ for a in an interval $]\lambda_1 - \delta, \lambda_1[$. In the right image one can see a curve $c \mapsto (a, t(c), c)$ for an a in $]\lambda_1 - \delta, \lambda_1[$ and one can see the boundary of the set $\Lambda \cap \{(t, c) : t \leq 0\}$. This figure corresponds to a case where $M > 0$.

4 Linear growth a between λ_1 and λ_2

In the case $c = 0$, equation (1) reduces to

$$-\Delta u = au - f(u). \quad (14)$$

From [10, Theorem 2.1], we know

Lemma 4.1. *Suppose that f satisfies (i)-(iv). The set of positive solutions (a, u) of (14) is a connected one dimensional manifold \mathcal{C}_+ of class C^1 in $\mathbb{R} \times \mathcal{H}$. The manifold is the union of the segment $\{(\lambda_1, t\phi) : t \in]0, M]\}$ with a graph $\{(a, u_\dagger(a)) : a \in]\lambda_1, +\infty[\}$. The solutions are strictly increasing along \mathcal{C}_+ . For $a > \lambda_1$ every positive solution is stable and, at each a , equation (14) has no other stable solution besides $u_\dagger(a)$.*

We turn to the case $c \neq 0$. Recall that by Remark 2.2, for a and c bounded, solutions (a, u, c) are such that u is bounded above. On the other hand, for c unbounded, we have

Lemma 4.2. *Let I be any compact interval in \mathbb{R} . There exists $K > 0$ such that, for all (a, u, c) solution of (1) with $a \in I$ and $|c| > K$, it holds $u \leq |c|$.*

Proof. By contradiction, suppose that (a_n, u_n, c_n) , with $a_n \in I$ and $|c_n| \rightarrow \infty$, is a sequence of solutions of (1), verifying that $\max u_n > |c_n|$. From Remark 2.2, (u_n) is uniformly bounded above, which is a contradiction. \square

Taking into account inequality (8), we immediately obtain

Corollary 4.3. *Under the conditions of the previous lemma, there exists a constant C such that $f(u) \leq C|c|$.*

In the following proposition we show that for large values of $|c|$ equation (1) has no solutions.

Proposition 4.4. *Let $\lambda_2 < a_2 < \lambda_3$ and $J =]\lambda_1, a_2]$. There exists $\bar{c} > 0$ such that for all $a \in J$ and (a, u, c) solution of (1), we have that $|c| \leq \bar{c}$.*

Proof. We argue by contradiction. Suppose that (a_n, u_n, c_n) is a solution to (1) with $a_n \in J$, and $c_n \rightarrow +\infty$ or $c_n \rightarrow -\infty$. We define $s = +1$ in the first case and $s = -1$ in the second case. Without loss of generality, we assume that $a_n \rightarrow a$. Define $v_n = \frac{u_n}{sc_n}$. The function v_n satisfies

$$\Delta v_n + a_n v_n - \frac{f(u_n)}{sc_n} - sh = 0. \quad (15)$$

From Corollary 4.3, we know that

$$\frac{f(u_n)}{sc_n} \leq C \quad (16)$$

for sufficiently large n . Since f is nonnegative, we also have $\frac{f(u_n)}{sc_n} \geq 0$. Recall that we assumed that λ_2 is simple and decompose

$$v_n = t_n \phi + \eta_n \psi + w_n,$$

where w_n denotes the component of v_n orthogonal to both ϕ and ψ . We prove successively that w_n is uniformly bounded in $L^2(\Omega)$ and that t_n, η_n are bounded. By Corollary 4.3, the function $\frac{f(u_n)}{sc_n} + sh$ is bounded in $L^\infty(\Omega)$. Thus, its component z_n orthogonal to the first two eigenfunctions is also bounded in $L^\infty(\Omega)$. Since a_n is bounded away from λ_3 , (w_n) is uniformly bounded in $L^2(\Omega)$. This is because

$$\Delta w_n + a_n w_n = z_n, \quad \text{or} \quad w_n = (\Delta + a_n)^{-1} z_n.$$

So, for example, the component of w_n on a third eigenfunction is the component of z_n on that third eigenfunction divided by $(a_n - \lambda_3)$. Similarly to (7), the values t_n are given by

$$(a_n - \lambda_1) t_n \int \phi^2 dx = \int \frac{f(u_n)}{sc_n} \phi dx$$

and hence are nonnegative, $t_n \geq 0$. An upper bound for t_n follows from $v_n \leq 1$, due to Lemma 4.2, which gives $t_n \leq \int \phi dx / \int \phi^2 dx$. Suppose that $|\eta_n| \rightarrow \infty$. The sequence $\eta_n \psi$ is not bounded above or below in $L^\infty(\Omega)$ because ψ changes sign. This contradicts $v_n \leq 1$ (because v_n^+ and w_n are bounded in $L^2(\Omega)$, $(\eta_n \psi)^+$ is unbounded in $L^2(\Omega)$ and $t_n \phi \geq 0$). We conclude (η_n) is bounded and (v_n) is uniformly bounded in $L^2(\Omega)$.

From (15), (v_n) is uniformly bounded in $H_0^1(\Omega)$. We may assume that $v_n \rightharpoonup v$ in $H_0^1(\Omega)$, $v_n \rightarrow v$ in $L^2(\Omega)$ and $v_n \rightarrow v$ a.e. in Ω . We claim that $v \leq 0$ a.e. in Ω . Suppose that there exists a point $x \in \Omega$, in the set where $v_n \rightarrow v$, such that $v(x) > 0$. Then $u_n(x) \rightarrow +\infty$. So,

$$\frac{f(u_n(x))}{sc_n} = \frac{f(u_n(x))}{u_n(x)} \frac{u_n(x)}{sc_n} \rightarrow +\infty \times v(x) = +\infty.$$

This contradicts (16) a.e. and shows that $v \leq 0$ a.e. in Ω . In order to pass to the limit in (15), we observe that $\frac{f(u_n)}{sc_n} \rightharpoonup f_\infty$ in $L^2(\Omega)$, where $f_\infty \geq 0$. The limit equation is

$$\Delta v + av - f_\infty - sh = 0.$$

Multiplying both sides by ϕ and integrating over Ω , we arrive to

$$(a - \lambda_1) \int v \phi dx = \int f_\infty \phi dx.$$

The left hand side is nonpositive and the right hand side is nonnegative. This implies that v and f_∞ are both identically equal to zero. We have reached a contradiction because h is nontrivial. \square

The conclusion of the previous proposition also holds for the case where λ_2 has multiplicity greater than one, since if a linear combination of second eigenfunctions is bounded, then each of the coefficients of that linear combination is bounded.

In the next lemma, we give a condition which ensures that a sequence of solutions of (1) converges, modulo a subsequence.

Lemma 4.5. *Let (a_n, u_n, c_n) be a sequence of solutions of (1) with $a_n \in J$, $a_n \rightarrow a$ and $c_n \rightarrow c$. Then, modulo a subsequence, (u_n) converges in \mathcal{H} .*

Proof. Using an argument similar to, but simpler than, the one in the proof of the previous proposition, one can show that (u_n) is uniformly bounded in $H_0^1(\Omega)$. By elliptic regularity theory, (u_n) is uniformly bounded in \mathcal{H} . Modulo a subsequence, $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, $u_n \rightarrow u$ in $L^2(\Omega)$ and $u_n \rightarrow u$ a.e. in Ω . Working with that subsequence and subtracting the equations for u_m and u_n , one can prove that (u_n) converges in \mathcal{H} . \square

Let $a > \lambda_1$. Starting at the stable solution $(a, u_\dagger(a), 0)$, and keeping a fixed, we can use the Implicit Function Theorem to follow a branch of solutions, taking c as parameter. Lemma 4.5 guarantees that the branch will not go to infinity. Since, from Proposition 4.4, solutions do not exist for large $|c|$, there must exist at least two degenerate solutions with Morse index equal to zero, $(a, u_*^-(a), c_*^-(a))$ and $(a, u_*^+(a), c_*^+(a))$, the first corresponding to a negative value of c and the second corresponding to a positive value of c . We recall that the Morse index of a solution is the number of negative eigenvalues of the linearized problem at the solution, and we recall that the solution is said to be degenerate if one of the eigenvalues of the linearized problem is equal to zero. In the next lemma we examine the behavior of the branch of solutions around a degenerate solution with Morse index equal to zero.

Lemma 4.6 ([8, Theorem 3.2], [13, p. 3613]). *Let $a > \lambda_1$ be fixed and $\mathbf{p}_* = (u_*, c_*)$ be a degenerate solution with Morse index equal to zero, with $c_* > 0$ (respectively $c_* < 0$). There exists a neighborhood of \mathbf{p}_* in $\mathcal{H} \times \mathbb{R}$ such that the set of solutions of (1) in the neighborhood is a C^1 manifold. This manifold is $\mathbf{m}^\# \cup \{\mathbf{p}_*\} \cup \mathbf{m}^*$. Here*

- $\mathbf{m}^\#$ is a manifold of nondegenerate solutions with Morse index equal to one, which is a graph $\{(u^\#(c), c) : c \in]c_* - \varepsilon_*, c_*[\} \cup \{(u^\#(c), c) : c \in]c_*, c_* + \varepsilon_*[\}$.
- \mathbf{m}^* is a manifold of stable solutions, which is a graph $\{(u^*(c), c) : c \in]c_* - \varepsilon_*, c_*[\} \cup \{(u^*(c), c) : c \in]c_*, c_* + \varepsilon_*[\}$.

The value ε_* is positive. The manifolds \mathbf{m}^\sharp and \mathbf{m}^* are connected by $\{\mathbf{p}_*\}$.

Proof. Let (u_*, c_*) be a degenerate solution with Morse index equal to zero. Let t_* and y_* be such that $u_* = t_* w_* + y_*$, with

$$w_* \in S := \{w \in \mathcal{H} : \int w^2 dx = \int \phi^2 dx\}$$

satisfying

$$\Delta w_* + a w_* - f'(u_*) w_* = 0. \quad (17)$$

$w_* > 0$, and

$$y_* \in \mathcal{R}_{w_*} = \{\omega \in \mathcal{H} : \int \omega w_* dx = 0\}.$$

We can assume that the first eigenfunction w_* is non negative because it minimizes the Rayleigh quotient. By Remark 2.2 and elliptic regularity theory, w_* is $C^2(\Omega)$. Suppose that w_* is zero at a point in the interior of Ω . As w_* has to be positive somewhere, centering a ball at a point where w_* is positive and enlarging it, one can pick a ball such that w_* is positive in the interior of the ball and zero somewhere on the boundary of the ball, say x_0 , with x_0 in the interior of Ω . Applying Lemma 3.4 of [9] (note the last assertion, if $w_*(x_0) = 0$ the same conclusion holds irrespective of the sign of the function c in [9]), we obtain that $\frac{\partial w_*}{\partial \nu}(x_0) < 0$ and this leads to w_* negative along the normal direction to the boundary of the ball, in the exterior of the ball. This contradicts that w_* is nonnegative. Hence w_* is positive in Ω .

Combining (1) with (17), we obtain that

$$\int (f'(u_*)u_* - f(u_*))w_* dx = c_* \int h w_* dx.$$

Observe that

$$\int h w_* dx \neq 0. \quad (18)$$

Indeed, otherwise $f'(u_*)u_* - f(u_*) = 0$. This implies that $u_* \leq M$, because of hypotheses (i)–(iii) on the function f . In fact, by (ii) and (iii), $u \mapsto f'(u)u - f(u)$ is increasing for $u \geq M$. Since the solution of the ordinary differential equation $f'(u)u - f(u) = 0$ is $f(u) = cu$ and f is continuous, $f'(u)u - f(u)$ cannot be zero for $u > M$ unless $M = 0$. But then f would not be differentiable at zero, contradicting (i). So $f'(u_*)u_* - f(u_*) = 0$ implies that $u_* \leq M$, as stated above. In this situation the term $f'(u_*)w_*$ in equation (17) would vanish and so $a = \lambda_1$, contrary to our assumption. Therefore, if $c_* > 0$, then $\int h w_* dx > 0$, and if $c_* < 0$, then $\int h w_* dx < 0$.

We let $\tilde{G} : \mathbb{R} \times \mathcal{R}_{w_*} \times \mathbb{R} \times S \times \mathbb{R} \rightarrow L^p(\Omega) \times L^p(\Omega)$ be defined by

$$\begin{aligned} \tilde{G}(t, y, c, w, \mu) &= (\Delta(tw_* + y) + a(tw_* + y) - f(tw_* + y) - ch, \\ &\quad \Delta w + aw - f'(tw_* + y)w + \mu w). \end{aligned}$$

We have that $\tilde{G}(t_*, y_*, c_*, w_*, 0) = 0$. We may use the Implicit Function Theorem to describe the solutions of $\tilde{G} = 0$ in a neighborhood of $(t_*, y_*, c_*, w_*, 0)$. Indeed, at this point,

$$\begin{aligned} \tilde{G}_y z + \tilde{G}_c \gamma + \tilde{G}_w \omega + \tilde{G}_\mu \nu &= (\Delta z + a z - f'(t_* w_* + y_*) z - \gamma h, \\ &\Delta \omega + a \omega - f'(t_* w_* + y_*) \omega \\ &- f''(t_* w_* + y_*) z w_* + \nu w_*). \end{aligned}$$

If this derivative vanishes, then we get that $\gamma = 0$ (multiply by w_* and integrate) and then $z = 0$ (because $z \in \mathcal{R}_{w_*}$ and the first eigenvalue is simple). Since $z = 0$, this implies (using the same argument) that $\nu = 0$ and then $\omega = 0$ (because $\omega \in \mathcal{R}_{w_*}$). The derivative is a homeomorphism from $\mathcal{R}_{w_*} \times \mathbb{R} \times \mathcal{R}_{w_*} \times \mathbb{R}$ to $L^p(\Omega) \times L^p(\Omega)$. So the solutions of $\tilde{G} = 0$ in a neighborhood of $(t_*, y_*, c_*, w_*, 0)$ lie on a curve $t \mapsto (t, y(t), c(t), w(t), \mu(t))$. Differentiating $\tilde{G}(t, y(t), c(t), w(t), \mu(t)) = (0, 0)$ once with respect to t ,

$$\begin{aligned} \Delta z + a z - f'(t_* w_* + y_*) z - \gamma h &= -(\Delta w_* + a w_* - f'(t_* w_* + y_*) w_*) = 0, \\ \Delta \omega + a \omega - f'(t_* w_* + y_*) \omega - f''(t_* w_* + y_*) z w_* + \nu w_* &= f''(t_* w_* + y_*) w_*^2 \end{aligned}$$

Here $z = y'(t_*)$, $\gamma = c'(t_*)$, $\omega = w'(t_*)$ and $\nu = \mu'(t_*)$. Clearly, both γ and z vanish. This implies that

$$\nu = \frac{\int f''(t_* w_* + y_*) w_*^3 dx}{\int w_*^2 dx}.$$

As we just saw, it is impossible for $\max_{\Omega}(t_* w_* + y_*) \leq M$. Hence,

$$\mu'(t_*) > 0. \quad (19)$$

Differentiating the first equation in $\tilde{G}(t, y(t), c(t), w(t), \mu(t)) = (0, 0)$ twice with respect to t , at t_* ,

$$\begin{aligned} \Delta z' + a z' - f'(t_* w_* + y_*) z' - \gamma' h - f''(t_* w_* + y_*) z(w_* + z) &= \\ f''(t_* w_* + y_*) w_* (w_* + z). \end{aligned}$$

Here $z = y'$. This can be rewritten as

$$\begin{aligned} \Delta z' + a z' - f'(t_* w_* + y_*) z' - \gamma' h &= f''(t_* w_* + y_*) (w_* + z)^2 \\ &= f''(t_* w_* + y_*) w_*^2, \end{aligned}$$

as $z(t_*) = 0$. Multiplying by w_* and integrating,

$$c''(t_*) = - \frac{\int f''(t_* w_* + y_*) w_*^3 dx}{\int h w_* dx}.$$

This is formula (2.7) of [13]. So $c''(t_*)$ is negative if $c_* > 0$ and positive if $c_* < 0$. Suppose that $c_* > 0$ (respectively, $c_* < 0$). As t increases from t_* , $c(t)$ decreases (respectively, increases) and the solution becomes stable. So the “end” of \mathbf{m}^* coincides with the piece of curve parametrized by $t \mapsto (c(t), t w(t) + y(t))$, for t in a right neighborhood of t_* . A parametrization of \mathbf{m}^\sharp is obtained by taking t in a left neighborhood of t_* . \square

The next proposition guarantees that the degenerate solutions vary smoothly with a .

Proposition 4.7. *The set of degenerate solutions (a, u, c) of (1) with Morse index equal to zero in $] \lambda_1, +\infty[\times \mathcal{H} \times \mathbb{R}$ is the disjoint union of two connected one dimensional manifolds \mathcal{D}_*^- and \mathcal{D}_*^+ of class C^1 . Each manifold is a graph $\{(a, u_*(a), c_*(a)) : a \in] \lambda_1, +\infty[\}$ and $\{(a, u_*(a), c_*(a)) : a \in] \lambda_1, +\infty[\}$.*

Sketch of the proof. To prove that the degenerate solutions can be followed using the parameter a , we apply (18) and the argument in the proof of Theorem 3.1 in [10]. On the other hand, suppose that there were more than the two degenerate solutions with Morse index equal to zero, $(a, u_*(a), c_*(a))$ and $(a, u_*(a), c_*(a))$, for each value of a . Then, because of Lemma 4.6, each additional degenerate solution would give rise to a branch of stable solutions, which could be followed using the parameter c , to $c = 0$. However, from Lemma 4.1, at $c = 0$ there exists only one stable solution $(a, u_+(a), 0)$. This would yield a contradiction. \square

We restrict our attention to $\lambda_1 < a < \lambda_2$. We observe that there are no degenerate solutions with Morse index greater than zero. Otherwise, we would have that the integral $\int [|\nabla v|^2 - av^2 + f'(u)v^2] dx$, and hence $\int [|\nabla v|^2 - av^2] dx$, is nonpositive on a two dimensional subspace of \mathcal{H} , which is not possible for $a < \lambda_2$. This observation and the above results lead to

Theorem 4.8. *Let $\lambda_1 < a < \lambda_2$. The set of solutions of (1) is a compact connected one dimensional manifold in $\{a\} \times \mathcal{H} \times \mathbb{R}$. There exist precisely two solutions for each $c \in]c_*^-, c_*^+[$, one stable, $(a, u_a^*(c), c)$, and the other nondegenerate with Morse index equal to one, $(a, u_a^\sharp(c), c)$. In addition, there exists exactly one degenerate solution with Morse index equal to zero when $c = c_*^-$ and $c = c_*^+$.*

In the next results, we consider the case $M > 0$. Recall the definitions of Λ in (2) and of T in (3). In parallel to Lemma 3.1, we can prove

Lemma 4.9. *Suppose $M > 0$. For $0 < \hat{t} < T$, define*

$$\Lambda_{\hat{t}} = \Lambda \cap \{(t, c) \in \mathbb{R}^2 : \hat{t} \leq t\} \quad \text{and} \quad \Lambda_{\hat{t}}^C = \{(t, c) \in \mathbb{R}^2 : t \geq 0\} \setminus \Lambda_{\hat{t}}$$

(see Figure 5). *There exists $\delta > 0$ such that for all $\lambda_1 < a < \lambda_1 + \delta$ and (a, u, c) solution of (1), we have $(t, c) \in \Lambda_{\hat{t}}^C$.*

The proof is analogous to the one of Lemma 3.1, but here we use Proposition 4.4 to guarantee that c is bounded and the fact that the last term in (11) is bounded, since (u_n) is uniformly bounded above, according to Remark 2.2.

We examine the behavior of \mathcal{D}_*^- and \mathcal{D}_*^+ as a decreases to λ_1 .

Proposition 4.10. *Suppose $M > 0$. As a decreases to λ_1 , we have*

$$\lim_{a \searrow \lambda_1} (a, u_*(a), c_*(a)) = (\lambda_1, 0 \phi + c_{*, \lambda_1}^-(\Delta + \lambda_1)^{-1} h, c_{*, \lambda_1}^-), \quad (20)$$

$$\lim_{a \searrow \lambda_1} (a, u_*(a), c_*(a)) = (\lambda_1, 0 \phi + c_{*, \lambda_1}^+(\Delta + \lambda_1)^{-1} h, c_{*, \lambda_1}^+), \quad (21)$$

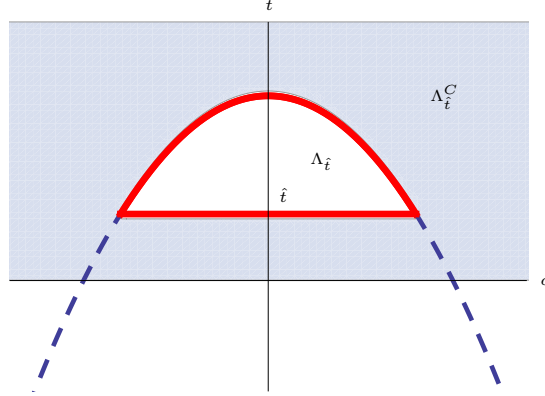


Figure 5: The regions $\Lambda_{\hat{t}}$ and $\Lambda_{\hat{t}}^C$.

i.e.

$$\lim_{a \searrow \lambda_1} (t_*^-(a), c_*^-(a)) = (0, c_{*,\lambda_1}^-), \quad \lim_{a \searrow \lambda_1} (t_*^+(a), c_*^+(a)) = (0, c_{*,\lambda_1}^+),$$

where $t_*^\pm(a) := \frac{\int u_*^\pm(a) \phi dx}{\int \phi^2 dx}$ and c_{*,λ_1}^- and c_{*,λ_1}^+ are given in (6).

Proof. By Proposition 4.4, $c_*^+(a)$ is bounded. By Lemma 4.5, we may assume, as $a \searrow \lambda_1$, that $(a, u_*^+(a), c_*^+(a))$ converges, say to $(\lambda_1, u_*^+(\lambda_1), c_*^+(\lambda_1))$, a solution of (1). From equality (7), $t_*^+(\lambda_1) := \frac{\int u_*^+(\lambda_1) \phi dx}{\int \phi^2 dx}$ is nonnegative. It is enough to prove that $c_*^+(\lambda_1) = c_{*,\lambda_1}^+ = c_{\lambda_1}^+(0)$ because if $(t, c_{\lambda_1}^+(0)) \in \Lambda$ and $t \geq 0$, then $t = 0$. Since $t_*^+(\lambda_1) \geq 0$ and c_*^+ is strictly decreasing, $c_*^+(\lambda_1) \leq c_{\lambda_1}^+(0)$. Suppose, by contradiction, that $\delta = c_{\lambda_1}^+(0) - c_*^+(\lambda_1) > 0$ (see Figure 6). According to the definition of $c_*^+(\lambda_1)$, there exists $\varepsilon > 0$ such that for all $\lambda_1 < a < \lambda_1 + \varepsilon$, we have that $c_*^+(a) < c_*^+(\lambda_1) + \delta/2$. Lemma 4.6 implies that for all $\lambda_1 < a < \lambda_1 + \varepsilon$ and (a, u, c) solution of (1), $c \leq c_*^+(a) < c_*^+(\lambda_1) + \delta/2$. Now choose $(t_0, c_0) \in \partial\Lambda$, with $t_0 > 0$ and $c_0 > c_{\lambda_1}^+(0) - \delta/2$. Applying Lemma 2.1 at (t_0, c_0) , we obtain solutions in a neighborhood of (t_0, c_0) corresponding to values of a close to λ_1 . We reach a contradiction to $c_*^+(a) < c_*^+(\lambda_1) + \delta/2$ for a in a right neighborhood of λ_1 . This proves $c_*^+(\lambda_1) = c_{\lambda_1}^+(0)$. \square

To conclude the analysis of the case $M > 0$, we state a result on uniform convergence of the curves of solutions, for fixed $\lambda_1 < a < \lambda_2$, as a decreases to λ_1 . We take into account that the set of values c for which there exists a solution, $[c_*^-(a), c_*^+(a)]$, in general depends on a . Of course, due to (20) and (21), for each $c_{*,\lambda_1}^- < c < c_{*,\lambda_1}^+$ there exist solutions (a, u, c) for that value of c and for a in a right neighborhood of λ_1 , while for each $c < c_{*,\lambda_1}^-$ and $c > c_{*,\lambda_1}^+$ there do not exist solutions (a, u, c) for that value of c and for a in a right neighborhood of λ_1 .

Theorem 4.11. Suppose $M > 0$ and let T be as in (3). When $c_{*,\lambda_1}^- \leq c \leq c_{*,\lambda_1}^+$, define t_c

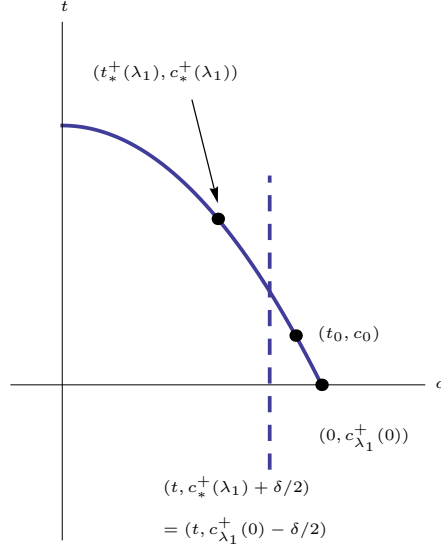


Figure 6: Illustration for the proof of Proposition 4.10.

by

$$t_c = \begin{cases} (c_{\lambda_1}^-)^{-1}(c) & \text{if } c < c_{\lambda_1}^-(T), \\ T & \text{if } c_{\lambda_1}^-(T) \leq c \leq c_{\lambda_1}^+(T), \\ (c_{\lambda_1}^+)^{-1}(c) & \text{if } c > c_{\lambda_1}^+(T). \end{cases}$$

For all $\delta > 0$, there exists $\varepsilon > 0$ satisfying for all $\lambda_1 < a < \lambda_1 + \varepsilon$ if the value c is such that there exists a solution $(a, u_a(c), c)$ then, in the case $c < c_{*, \lambda_1}^-$ we have

$$\|u_a(c) - (0\phi + c_{*, \lambda_1}^-(\Delta + \lambda_1)^{-1}h)\|_{\mathcal{H}} < \delta,$$

and in the case $c > c_{*, \lambda_1}^+$ we have

$$\|u_a(c) - (0\phi + c_{*, \lambda_1}^+(\Delta + \lambda_1)^{-1}h)\|_{\mathcal{H}} < \delta;$$

if the value c is such that there exist solutions $(a, u_a^*(c), c)$ and $(a, u_a^\sharp(c), c)$ then, in the case $c_{*, \lambda_1}^- \leq c \leq c_{*, \lambda_1}^+$, we have

$$\begin{aligned} \|u_a^*(c) - (t_c\phi + c(\Delta + \lambda_1)^{-1}h)\|_{\mathcal{H}} &< \delta, \\ \|u_a^\sharp(c) - (0\phi + c(\Delta + \lambda_1)^{-1}h)\|_{\mathcal{H}} &< \delta. \end{aligned}$$

The proof is similar to the one of Theorem 3.2.

In Figure 7, we plot the curves $c \mapsto (a, t(c), c)$ for a in an interval $]\lambda_1, \lambda_1 + \delta[$, for some small $\delta > 0$ and $M > 0$.

To conclude this section we consider the case $M = 0$.

Theorem 4.12. *Suppose $M = 0$. For all $\delta > 0$, there exists $\varepsilon > 0$ satisfying for all $\lambda_1 < a < \lambda_1 + \varepsilon$ if the value c is such that there exists a solution $(a, u_a(c), c)$, then*

$$\|u_a(c)\|_{\mathcal{H}} < \delta.$$

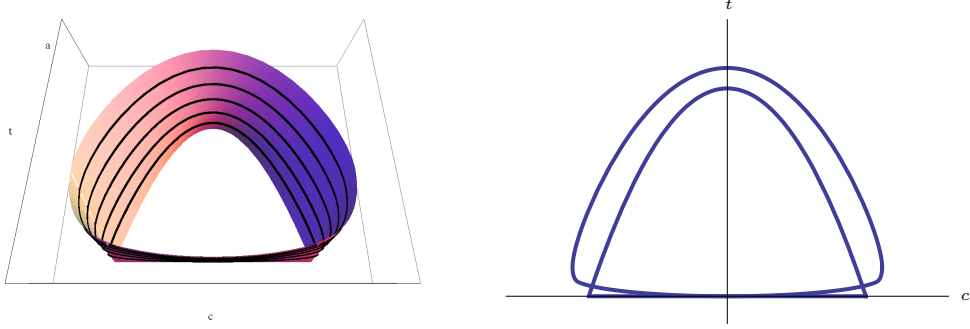


Figure 7: In the left image one can see the surface formed by the curves $c \mapsto (a, t(c), c)$ for a in an interval $]\lambda_1, \lambda_1 + \delta[$. In the right image one can see a curve $c \mapsto (a, t(c), c)$ for an a in $]\lambda_1, \lambda_1 + \delta[$ and one can see the boundary of the set $\Lambda \cap \{(t, c) : t \geq 0\}$. This figure corresponds to a case where $M > 0$.

5 Linear growth a greater than or equal to λ_2

We recall the assumption that the second eigenvalue of the Dirichlet Laplacian on Ω is simple and we call ψ an associated eigenfunction, normalized so $\max_{\Omega} \psi = 1$. We define

$$\beta = -\min_{\Omega} \psi,$$

so that $\beta > 0$. To fix ideas, without loss of generality, we suppose

$$\int h\psi \, dx < 0. \quad (22)$$

For $a = \lambda_2$, the set of solutions of (1) is completely described by

Theorem 5.1. *Suppose f satisfies (i)-(iv) and h satisfies (a)-(c). Fix $a = \lambda_2$. The set of solutions (λ_2, u, c) of (1) is a compact connected one dimensional manifold \mathcal{M} of class C^1 in $\{\lambda_2\} \times \mathcal{H} \times \mathbb{R}$. We have*

$$\mathcal{M} = \mathcal{M}^b \cup \mathcal{L} \cup \mathcal{M}^\sharp \cup \{\mathbf{p}_*^+\} \cup \mathcal{M}^* \cup \{\mathbf{p}_*^-\},$$

where \mathcal{L} connects \mathcal{M}^b and \mathcal{M}^\sharp , $\{\mathbf{p}_*^+\}$ connects \mathcal{M}^\sharp and \mathcal{M}^* , and, finally, $\{\mathbf{p}_*^-\}$ connects \mathcal{M}^* and \mathcal{M}^b . Here

- \mathcal{M}^b is a manifold of nondegenerate solutions with Morse index equal to one, which is a graph $\{(\lambda_2, u_{\lambda_2}^b(c), c) : c \in]c_*^-(\lambda_2), 0[\}$.
- \mathcal{L} is a segment (a point in the case $M = 0$) of degenerate solutions with Morse index equal to one, $\{(\lambda_2, t\psi, 0) : t \in [-\frac{M}{\beta}, M] \}$.
- \mathcal{M}^\sharp is a manifold of nondegenerate solutions with Morse index equal to one, which is a graph $\{(\lambda_2, u_{\lambda_2}^\sharp(c), c) : c \in]0, c_*^+(\lambda_2)[\}$.

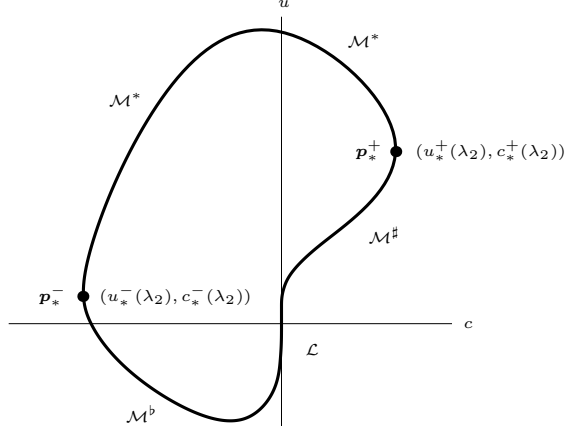


Figure 8: A bifurcation curve for $a = \lambda_2$.

- $\mathbf{p}_*^+ = (\lambda_2, u_*^+(\lambda_2), c_*^+(\lambda_2))$ is a degenerate solution with Morse index equal to zero.
- \mathcal{M}^* is a manifold of stable solutions, which is a graph $\{(\lambda_2, u_{\lambda_2}^*(c), c) : c \in]c_*^-(\lambda_2), c_*^+(\lambda_2)[\}$.
- $\mathbf{p}_*^- = (\lambda_2, u_*^-(\lambda_2), c_*^-(\lambda_2))$ is a degenerate solution with Morse index equal to zero.

Theorem 5.1 is illustrated in Figure 8.

Sketch of the proof. We start at $(\lambda_2, u_\dagger(\lambda_2), 0)$, the stable solution at $c = 0$. We may use the Implicit Function Theorem to follow the solutions for $c \in]c_*^-(\lambda_2), c_*^+(\lambda_2)[$, arriving at the left at \mathbf{p}_*^- and at the right at \mathbf{p}_*^+ . Arguing as in the proof of Theorem 1.2 of [10], \mathbf{p}_*^+ is connected successively to \mathcal{M}^\sharp , \mathcal{L} and \mathcal{M}^\flat . By Lemma 6.1 in [10], we may use the parameter c to follow the branch \mathcal{M}^\flat until we reach a degenerate solution with Morse index equal to zero, as all solutions with Morse index equal to one are nondegenerate, except for $(\lambda_2, t\psi, 0)$ for $t \in [-\frac{M}{\beta}, M]$. The degenerate solution with Morse index equal to zero must be \mathbf{p}_*^- , since it is the only one corresponding to a negative value of c . The branches \mathcal{M}^\flat and \mathcal{M}^* connect at \mathbf{p}_*^- . This is in accordance to Lemma 4.6. \square

When $a > \lambda_2$, the set of solutions of (1) is characterized in

Theorem 5.2. *Suppose f satisfies (i)-(iv) and h satisfies (a)-(c). Without loss of generality, suppose (22) is true. There exists $\delta > 0$ such that the following holds. Fix $\lambda_2 < a < \lambda_2 + \delta$. The set of solutions (a, u, c) of (1) is a compact connected one dimensional manifold \mathcal{M} of class C^1 in $\{a\} \times \mathcal{H} \times \mathbb{R}$. We have \mathcal{M} is the disjoint union*

$$\mathcal{M} = \mathcal{M}^\flat \cup \{\mathbf{p}_\flat\} \cup \mathcal{M}^\sharp \cup \{\mathbf{p}_\sharp\} \cup \mathcal{M}^\# \cup \{\mathbf{p}_*^+\} \cup \mathcal{M}^* \cup \{\mathbf{p}_*^-\},$$

where $\{\mathbf{p}_\flat\}$ connects \mathcal{M}^\flat and \mathcal{M}^\sharp , $\{\mathbf{p}_\sharp\}$ connects \mathcal{M}^\sharp and $\mathcal{M}^\#$, $\{\mathbf{p}_*^+\}$ connects $\mathcal{M}^\#$ and \mathcal{M}^* , and $\{\mathbf{p}_*^-\}$ connects \mathcal{M}^* and \mathcal{M}^\flat . Here

- \mathcal{M}^\flat is a manifold of nondegenerate solutions with Morse index equal to one, which is a graph $\{(a, u_a^\flat(c), c) : c \in]c_*^-(a), c_\flat(a)[\}$.

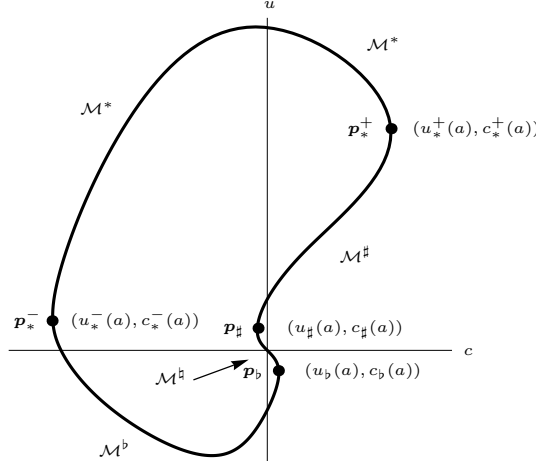


Figure 9: A bifurcation curve for $\lambda_2 < a < \lambda_2 + \delta$.

- $p_b = (a, u_b(a), c_b(a))$ is a degenerate solution with Morse index equal to one.
- \mathcal{M}^b is a manifold of solutions with Morse index equal to one or to two,

$$\{(a, u_a^b(t), c_a^b(t)) : u_a^b(t) = t\psi + y_a^b(t), t \in J\},$$

with $c_a^b : J \rightarrow \mathbb{R}$, $y_a^b : J \rightarrow \{y \in \mathcal{H} : \int y\psi dx = 0\}$ and $J =]-\frac{M}{\beta} - \varepsilon_b, M + \varepsilon_\#[$, for some $\varepsilon_b, \varepsilon_\# > 0$.

- $p_\# = (a, u_\#(a), c_\#(a))$ is a degenerate solution with Morse index equal to one.
- $\mathcal{M}^\#$ is a manifold of nondegenerate solutions with Morse index equal to one, which is a graph $\{(a, u_a^\#(c), c) : c \in]c_\#(a), c_*^+(a)[\}$.
- $p_*^+ = (a, u_*^+(a), c_*^+(a))$ is a degenerate solution with Morse index equal to zero.
- \mathcal{M}^* is the manifold of stable solutions, which is a graph $\{(a, u_a^*(c), c) : c \in]c_*^-, (a), c_*^+(a)[\}$.
- $p_*^- = (a, u_*^-(a), c_*^-(a))$ is a degenerate solution with Morse index equal to zero.

We have $(c_a^b)'(0) < 0$ and

$$\begin{aligned} \lim_{t \searrow -\frac{M}{\beta} - \varepsilon_b} (a, u_a^b(t), c_a^b(t)) &= (a, u_b(a), c_b(a)), \\ \lim_{t \nearrow M + \varepsilon_\#} (a, u_a^b(t), c_a^b(t)) &= (a, u_\#(a), c_\#(a)). \end{aligned}$$

In particular, if $|c|$ is sufficiently small, then (1) has at least four solutions.

Theorem 5.2 is illustrated in Figure 9.

Sketch of the proof. Using Lemma 7.3 of [10], we introduce a chart $]\lambda_2 - \varepsilon, \lambda_2 + \varepsilon[\times]-\frac{M}{\beta} - \tilde{\varepsilon}, M + \tilde{\varepsilon}[$ in the (a, t) plane, around $(\lambda_2, 0)$, to describe the solutions of (1) in a neighborhood of $(\lambda_2, 0, 0)$. Suppose $\lambda_2 < a < \lambda_2 + \delta$. We start at $(a, 0)$ and vary the parameter t in $]-\frac{M}{\beta} - \tilde{\varepsilon}, M + \tilde{\varepsilon}[$ to follow the solutions of (1). By choosing δ small enough, we can guarantee that when t reaches $-\frac{M}{\beta} - \tilde{\varepsilon}$ and $M + \tilde{\varepsilon}$ we can switch to the parameter c to follow the solutions of (1). Indeed, consider the curve \mathcal{D}_ζ of degenerate solutions with Morse index equal to one, constructed in Lemma 7.2 of [10]. A small enough choice of δ will make the projection of \mathcal{D}_ζ in the chart not intersect either $(a, -\frac{M}{\beta} - \tilde{\varepsilon})$ or $(a, M + \tilde{\varepsilon})$. Arguing as in the proof of Proposition 7.5 of [10], we know that the solutions with coordinates $(a, -\frac{M}{\beta} - \tilde{\varepsilon})$ and $(a, M + \tilde{\varepsilon})$ are nondegenerate and have Morse index equal to one. We also know that when we arrive at the solution with coordinates $(a, M + \tilde{\varepsilon})$ we have to increase c , and when we arrive at the solution with coordinates $(a, -\frac{M}{\beta} - \tilde{\varepsilon})$ we have to decrease c , to follow the solutions out of the chart. By further reducing δ , if necessary, Lemma 7.4 of [10] together with Proposition 4.4, assure that there are no degenerate solutions with Morse index equal to one when we use the parameter c to follow the solutions outside the chart. If we find a degenerate solution it will necessarily have Morse index equal to zero. But we know these lie on \mathcal{D}_*^- and \mathcal{D}_*^+ . So one can finish by arguing as in the proof of Theorem 5.1. \square

We finish with a word on qualitative properties of solutions. A nodal domain of a function u is a connected component of $\Omega \setminus u^{-1}(0)$. There are several works relating the number of nodal domains of solutions of elliptic equations with their Morse indices. Possibly the first result in this direction was the Courant Nodal Domain Theorem [6], which states that number of nodal domains of an n -th eigenfunction of the Dirichlet Laplacian on Ω is less than or equal to n . But Courant gave examples where the n -th eigenfunction only has two nodal domains.

In the one dimensional case, $\Omega \subset \mathbb{R}^N$ with $N = 1$, Sturm's Comparison Theorems allow one to establish that for some *linear homogeneous* equations the number of nodal domains of a solution is equal to its Morse index (see [4, Chapter 8, Theorem 2.1]).

Courant's Nodal Domain Theorem was also generalized to some superlinear elliptic equations (for example, see [1], [2] and [15]). Let us recall, in general terms, how the main argument goes. Suppose $u \in H_0^1(\Omega)$ is a weak solution of

$$-\Delta u = g(x, u) \quad \text{in } \Omega, \quad (23)$$

where g satisfies appropriate growth conditions, $g(x, 0) = 0$ for all $x \in \Omega$, and $g'(x, t)t^2 > g(x, t)t$ for all $x \in \Omega$ and $t \neq 0$ (where the prime denotes the derivative with respect to t). Let $v = u\chi$, where χ is the characteristic function of a nodal domain of u . Using [12, Lemma 1], the function v belongs to $H_0^1(\Omega)$. Multiplying both sides of (23) by v and integrating over Ω , and using the superlinear assumption on g , one obtains

$$\int |\nabla v|^2 dx = \int g(x, v)v dx < \int g'(x, v)v^2 dx.$$

Thus, the functional

$$v \mapsto \int |\nabla v|^2 dx - \int g'(x, u)v^2 dx$$

is negative in the direction v . Furthermore, two different nodal domains are associated to two independent functions v , with disjoint supports. So the Morse index of u is greater than or equal to the number of nodal domains of u .

The type of argument in the previous paragraph does not apply to (1) where

$$g(x, u) = au - f(u) - ch(x)$$

is not superlinear and does not vanish at $u = 0$, unless $c = 0$. Moreover, using specific examples, in the simplest one dimensional case, one can check that a stable a solution of (1) may have two nodal domains. Hence, the usual relation between the number of nodal domains and the Morse index does not hold for our equation.

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